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Pfaffian and determinant solutions to a discretized Toda equation for B_r, C_r and D_r

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Abstract. We consider a class of two-dimensional Toda equations on discrete spacetime. It has arisen as functional relations in a commuting family of transfer matrices in solvable lattice models associated with any classical simple Lie algebra X_r. For $X_r = B_r, C_r$ and D_r, we present the solution in terms of Pfaffians and determinants. They may be viewed as Yangian analogues of the classical Jacobi–Trudi formula on Schur functions.

1. Introduction

Consider the following systems of difference equations on $T_m^{(a)}(u)$ $(m \in \mathbb{Z}_{\geqslant 0}, u \in \mathbb{C}, a \in \{1, 2, \ldots, r\})$:

$B_r : (r \geqslant 2)$

$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u) \qquad 1 \leqslant a \leqslant r-2 \tag{1a}$$

$$T_m^{(r-1)}(u-1)T_m^{(r-1)}(u+1) = T_{m+1}^{(r-1)}(u)T_{m-1}^{(r-1)}(u) + T_m^{(r-2)}(u)T_{2m}^{(r)}(u) \tag{1b}$$

$$T_{2m}^{(r)}(u-\tfrac{1}{2})T_{2m}^{(r)}(u+\tfrac{1}{2}) = T_{2m+1}^{(r)}(u)T_{2m-1}^{(r)}(u) + T_m^{(r-1)}(u-\tfrac{1}{2})T_m^{(r-1)}(u+\tfrac{1}{2}) \tag{1c}$$

$$T_{2m+1}^{(r)}(u-\tfrac{1}{2})T_{2m+1}^{(r)}(u+\tfrac{1}{2}) = T_{2m+2}^{(r)}(u)T_{2m}^{(r)}(u) + T_m^{(r-1)}(u)T_{m+1}^{(r-1)}(u)\,. \tag{1d}$$

$C_r : (r \geqslant 2)$

$$T_m^{(a)}(u-\tfrac{1}{2})T_m^{(a)}(u+\tfrac{1}{2}) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u) \qquad 1 \leqslant a \leqslant r-2 \tag{2a}$$

$$T_{2m}^{(r-1)}(u-\tfrac{1}{2})T_{2m}^{(r-1)}(u+\tfrac{1}{2}) = T_{2m+1}^{(r-1)}(u)T_{2m-1}^{(r-1)}(u) + T_{2m}^{(r-2)}(u)T_m^{(r)}(u-\tfrac{1}{2})T_m^{(r)}(u+\tfrac{1}{2}) \tag{2b}$$

$$T_{2m+1}^{(r-1)}(u-\tfrac{1}{2})T_{2m+1}^{(r-1)}(u+\tfrac{1}{2}) = T_{2m+2}^{(r-1)}(u)T_{2m}^{(r-1)}(u) + T_{2m+1}^{(r-2)}(u)T_m^{(r)}(u)T_{m+1}^{(r)}(u) \tag{2c}$$

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$$T^{(r)}_m(u-1)T^{(r)}_m(u+1) = T^{(r)}_{m+1}(u)T^{(r)}_{m-1}(u) + T^{(r-1)}_{2m}(u). \tag{2d}$$

$D_r : (r \geqslant 4)$

$$T^{(a)}_m(u-1)T^{(a)}_m(u+1) = T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + T^{(a-1)}_m(u)T^{(a+1)}_m(u) \qquad 1 \leqslant a \leqslant r-3 \tag{3a}$$

$$T^{(r-2)}_m(u-1)T^{(r-2)}_m(u+1) = T^{(r-2)}_{m+1}(u)T^{(r-2)}_{m-1}(u) + T^{(r-3)}_m(u)T^{(r-1)}_m(u)T^{(r)}_m(u) \tag{3b}$$

$$T^{(a)}_m(u-1)T^{(a)}_m(u+1) = T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + T^{(r-2)}_m(u) \qquad a = r-1, r. \tag{3c}$$

($T^{(0)}_m(u) = 1$.) We shall consider the initial condition $T^{(a)}_0(u) = 1$ for any $1 \leqslant a \leqslant r$ exclusively. Then one can solve the systems (1), (2) and (3) iteratively to express $T^{(a)}_m(u)$ in terms of $T^{(1)}_1(u+\text{shift}), \ldots, T^{(r)}_1(u+\text{shift})$. For example, $T^{(1)}_2(u) = T^{(1)}_1(u-1)T^{(1)}_1(u+1) - T^{(2)}_1(u)$ from (1*a*). The purpose of this paper is to present the formulae that express an arbitrary $T^{(a)}_m(u)$ $(m \geqslant 1)$ as a determinant or a Pfaffian of matrices with elements 0 or $\pm T^{(b)}_1(u+\text{shift})$ $(0 \leqslant b \leqslant r)$.

In fact, such formulae had been partially conjectured in [KNS1], where a set of functional relations, T-system, was introduced for the commuting family of transfer matrices $\{T^{(a)}_m(u)\}$ for solvable lattice models associated with any classical simple Lie algebra X_r. In this context, equations (1), (2) and (3) correspond to $X_r = B_r, C_r$ and D_r cases of the T-system, respectively. $T^{(a)}_m(u)$ denotes a transfer matrix (or its eigenvalue) with spectral parameter u and 'fusion type' labelled by a and m [KNS1]. Our result here confirms all of the determinant conjectures raised in section 5 of [KNS1]. Moreover, it extends them to a full solution of (1), (2) and (3), which, in general, involves Pfaffians as well. In the representation theoretical viewpoint, this yields a Yangian analogue of the Jacobi–Trudi formula [Ma], i.e. a way of constructing Yangian characters from those for the fundamental representations [CP].

Beside the significance in the lattice model context [KNS2], the beautiful structure in these solutions indicate a rich content of the T-system also as an example of discretized soliton equations [AL, H1, H2, H3, H4, HTI, K, DJM, S, VF, BKP, Wa, Wi]. In fact, regarding u and m as continuous spacetime coordinates, one can take a suitable scaling limit where the T-system becomes a two-dimensional Toda (or Toda molecule) equation for X_r [T, MOP, LS]:

$$(\partial_u^2 - \partial_m^2)\ \log \phi_a(u,m) = \text{constant} \times \prod_{b=1}^{r} \phi_b(u,m)^{-A_{ab}}\ . \tag{4}$$

Here $\phi_a(u,m)$ is a scaled $T^{(a)}_m(u)$ and $A_{ab} = 2(\alpha_a|\alpha_b)/(\alpha_a|\alpha_a)$ is the Cartan matrix. In this sense, our T-system is a discretization of the Toda equation allowing determinant and Pfaffian solutions at least for $X_r = A_r, B_r, C_r$ and D_r. See also the remarks in section 6 concerning the T-systems for twisted affine Lie algebras [KS].

The outline of the paper is as follows. In sections 2, 3 and 4, we present solutions to the B_r, C_r and D_r cases, respectively. Pfaffians are needed for $T^{(r)}_m(u)$ in C_r and $T^{(r-1)}_m(u)$ and $T^{(r)}_m(u)$ in D_r. In section 5, we illustrate a proof for the C_r case. The other cases can be verified quite similarly. Section 6 is devoted to a summary and discussion.

Before closing the introduction, a few remarks are in order. Firstly, the original T-system [KNS1] had a factor $g^{(a)}_m(u)$ in front of the second term on the RHS of $T^{(a)}_m(u+(1/t_a))T^{(a)}_m(u-(1/t_a)) = \cdots$. Throughout this paper we shall set $g^{(a)}_m(u) = 1$. To recover the dependence on $g^{(a)}_m(u)$ is quite easy as long as the relation $g^{(a)}_m(u+(1/t_a))g^{(a)}_m(u-$

$(1/t_a)) = g^{(a)}_{m+1}(u)g^{(a)}_{m-1}(u)$ is satisfied (cf [KNS1]). Secondly, the T-systems (1) and (2) coincide for $r = 2$ under the exchange $T^{(1)}_m(u) \leftrightarrow T^{(2)}_m(u)$, which reflects the Lie algebra equivalence $B_2 \simeq C_2$. In this case equations (9) and (12) yield two alternative expressions for the same quantity. Thirdly, for $X_r = A_r$, the T-system $T^{(a)}_m(u-1)T^{(a)}_m(u+1) = T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + T^{(a-1)}_m(u)T^{(a+1)}_m(u)$ $(1 \leqslant a \leqslant r, T^{(0)}_m(u) = T^{(r+1)}_m(u) = 1)$ is the so-called Hirota–Miwa equation. In the transfer matrix context, it has been proved in [KNS1] by using the determinantal formula in [BR]. Finally, for $X_r = B_r$, a determinantal solution different from (9) has been obtained in [KOS]. The relevant matrix there is not sparse, unlike equations (7) and the matrix elements are not necessarily $T^{(a)}_1(u)$ but contain some quadratic expressions of $T^{(r)}_1(u)$ in general.

2. The B_r case

For any $k \in \mathbb{C}$, put

$$x^a_k = \begin{cases} T^{(a)}_1(u+k) & 1 \leqslant a \leqslant r \\ 1 & a = 0\,. \end{cases} \tag{5}$$

We introduce the infinite-dimensional matrices $\mathcal{T} = (\mathcal{T}_{ij})_{i,j\in\mathbb{Z}}$ and $\mathcal{E} = (\mathcal{E}_{ij})_{i,j\in\mathbb{Z}}$ as follows:

$$\mathcal{T}_{ij} = \begin{cases} x^{\frac{1}{2}(j-i)+1}_{\frac{1}{2}(i+j)-1} & \text{if } i \in 2\mathbb{Z}+1 \text{ and } \frac{1}{2}(i-j) \in \{1, 0, \ldots, 2-r\} \\ -x^{\frac{1}{2}(i-j)+2r-2}_{\frac{1}{2}(i+j)-1} & \text{if } i \in 2\mathbb{Z}+1 \text{ and } \frac{1}{2}(i-j) \in \{1-r, -r, \ldots, 2-2r\} \\ -x^r_{r+i-\frac{5}{2}} & \text{if } i \in 2\mathbb{Z} \text{ and } j = i+2r-3 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$\mathcal{E}_{ij} = \begin{cases} \pm 1 & \text{if } i = j-1\pm 1 \text{ and } i \in 2\mathbb{Z} \\ x^r_{i-1} & \text{if } i = j-1 \text{ and } i \in 2\mathbb{Z}+1 \\ 0 & \text{otherwise}\,. \end{cases}$$

For example, for B_3, they read

$$(\mathcal{T}_{ij})_{i,j\geqslant 1} = \begin{pmatrix} x^1_0 & 0 & x^2_1 & 0 & -x^2_2 & 0 & -x^1_3 & 0 & -1 & \\ 0 & 0 & 0 & 0 & -x^3_{\frac{5}{2}} & 0 & 0 & 0 & 0 & \\ 1 & 0 & x^1_2 & 0 & x^2_3 & 0 & -x^2_4 & 0 & -x^1_5 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -x^3_{\frac{9}{2}} & 0 & 0 & \\ 0 & 0 & 1 & 0 & x^1_4 & 0 & x^2_5 & 0 & -x^2_6 & \\ & & & & \vdots & & & & & \ddots \end{pmatrix} \tag{7a}$$

$$(\mathcal{E}_{ij})_{i,j\geqslant 1} = \begin{pmatrix} 0 & x^3_0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & x^3_2 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & x^3_4 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \\ & & & \vdots & & & & \ddots \end{pmatrix}. \tag{7b}$$

As is evident from the above example, for any $1 \leqslant a \leqslant r$ and k, the quantity $\pm x^a_k$ is contained in $\mathcal{T}|_{u\to u+\xi}$ once and only once as its matrix element. Here $u \to u+\xi$ means the overall shift of lower indices in accordance with (5). For example, the shift $\xi = 1$ is necessary to accommodate x^1_1 as the (1,1) element of $\mathcal{T}|_{u\to u+\xi}$. In view of this, we shall employ the notation $\mathcal{T}_m(i, j, \pm x^a_k)$ to mean the m by m submatrix of $\mathcal{T}|_{u\to u+\xi}$ whose (i, j) element is exactly $\pm x^a_k$. This definition is unambiguous irrespective of various possible choices of ξ. For example, in equation (7*a*),

$$\begin{gathered}\mathcal{T}_3(1,1,x^1_0) = \begin{pmatrix} x^1_0 & 0 & x^2_1 \\ 0 & 0 & 0 \\ 1 & 0 & x^1_2 \end{pmatrix} \qquad \mathcal{T}_3(1,1,x^1_1) = \begin{pmatrix} x^1_1 & 0 & x^2_2 \\ 0 & 0 & 0 \\ 1 & 0 & x^1_3 \end{pmatrix} \\ \mathcal{T}_2(1,2,-x^3_{\frac{5}{2}}) = \begin{pmatrix} 0 & -x^3_{\frac{5}{2}} \\ 0 & x^2_3 \end{pmatrix} \qquad \mathcal{T}_2(1,2,-x^3_2) = \begin{pmatrix} 0 & -x^3_2 \\ 0 & x^2_{\frac{5}{2}} \end{pmatrix}.\end{gathered} \tag{8}$$

We shall also use the similar notation $\mathcal{E}_m(i, j, \pm x^r_k)$. With this notation our result in this section is stated as follows.

Theorem 2.1. For $m \in \mathbb{Z}_{\geqslant 1}$

$$T^{(a)}_m(u) = \det\big(\mathcal{T}_{2m-1}(1,1,x^a_{-m+1}) + \mathcal{E}_{2m-1}(1,2,x^r_{-m+r-a+\frac{1}{2}})\big) \qquad 1 \leqslant a < r \tag{9a}$$

$$T^{(r)}_m(u) = (-1)^{m(m-1)/2} \det\big(\mathcal{T}_m(1,2,-x^{r-1}_{-\frac{1}{2}m+1}) + \mathcal{E}_m(1,1,x^r_{-\frac{1}{2}m+\frac{1}{2}})\big) \tag{9b}$$

solves the B_r T-system (1).

Up to some conventional change, equation (9*a*) in the above had been conjectured in equation (5.6) of [KNS1]. The formula (9*b*) is new.

3. The C_r case

Here we introduce the inifinite-dimensional matrix $\mathcal{T}$ by

$$\mathcal{T}_{ij} = \begin{cases} x^{j-i+1}_{\frac{1}{2}(i+j)-1} & \text{if } i-j \in \{1,0,\ldots,1-r\} \\ -x^{i-j+2r+1}_{\frac{1}{2}(i+j)-1} & \text{if } i-j \in \{-1-r,-2-r,\ldots,-1-2r\} \\ 0 & \text{otherwise}. \end{cases} \tag{10}$$

For example, for C_2, it reads

$$(\mathcal{T}_{ij})_{i,j\geqslant 1} = \begin{pmatrix} x^1_0 & x^2_{\frac{1}{2}} & 0 & -x^2_{\frac{3}{2}} & -x^1_2 & -1 & 0 & 0 & \\ 1 & x^1_1 & x^2_{\frac{3}{2}} & 0 & -x^2_{\frac{5}{2}} & -x^1_3 & -1 & 0 & \cdots \\ 0 & 1 & x^1_2 & x^2_{\frac{5}{2}} & 0 & -x^2_{\frac{7}{2}} & -x^1_4 & -1 & \\ 0 & 0 & 1 & x^1_3 & x^2_{\frac{7}{2}} & 0 & -x^2_{\frac{9}{2}} & -x^1_5 & \\ & & & & \vdots & & & & \ddots \end{pmatrix}. \tag{11}$$

We keep the same notation (5) and $\mathcal{T}_m(i, j, \pm x^a_k)$ $(1 \leqslant a \leqslant r)$ as in section 2. Note that $\mathcal{T}_m(1, 2, -x^r_k)$ is an antisymmetric matrix for any m. Our result in this section is stated as follows.

Theorem 3.1. For $m \in \mathbb{Z}_{\geqslant 1}$

$$T_m^{(a)}(u) = \det\big(\mathcal{T}_m(1, 1, x^a_{-\frac{1}{2}m+\frac{1}{2}})\big) \qquad 1 \leqslant a < r \tag{12a}$$

$$T_m^{(r)}(u) = (-1)^m \operatorname{pf}\big(\mathcal{T}_{2m}(1, 2, -x^r_{-m+1})\big) \tag{12b}$$

solves the C_r T-system (2).

The expression (12a) is essentially conjecture (5.10) in [KNS1]. The Pfaffian formula (12b) is new. In proving theorem 3.1 in section 5, we will also establish the relations

$$T_m^{(r)}(u - \tfrac{1}{2})T_m^{(r)}(u + \tfrac{1}{2}) = \det\big(\mathcal{T}_{2m}(1, 1, x^r_{-m+\frac{1}{2}})\big) \tag{13a}$$

$$T_m^{(r)}(u)T_{m+1}^{(r)}(u) = \det\big(\mathcal{T}_{2m+1}(1, 1, x^r_{-m})\big)\,. \tag{13b}$$

4. The D_r case

Here we define the infinite-dimensional matrices $\mathcal{T}$ and $\mathcal{E}$ by

$$\mathcal{T}_{ij} = \begin{cases} x^{\frac{1}{2}(j-i)+1}_{\frac{1}{2}(i+j)-1} & \text{if } i \in 2\mathbb{Z}+1 \text{ and } \frac{1}{2}(i-j) \in \{1, 0, \ldots, 3-r\} \\ -x^{r-1}_{\frac{1}{2}(i+j-1)} & \text{if } i \in 2\mathbb{Z}+1 \text{ and } \frac{1}{2}(i-j) = \frac{5}{2} - r \\ -x^{r}_{\frac{1}{2}(i+j-3)} & \text{if } i \in 2\mathbb{Z}+1 \text{ and } \frac{1}{2}(i-j) = \frac{3}{2} - r \\ -x^{\frac{1}{2}(i-j)+2r-3}_{\frac{1}{2}(i+j)-1} & \text{if } i \in 2\mathbb{Z}+1 \text{ and } \\ & \frac{1}{2}(i-j) \in \{1-r, -r, \ldots, 3-2r\} \\ 0 & \text{otherwise} \end{cases} \tag{14a}$$

$$\mathcal{E}_{ij} = \begin{cases} \pm 1 & \text{if } i = j - 2 \pm 2 \text{ and } i \in 2\mathbb{Z} \\ x^{r-1}_{i} & \text{if } i = j - 3 \text{ and } i \in 2\mathbb{Z} \\ x^{r}_{i-2} & \text{if } i = j - 1 \text{ and } i \in 2\mathbb{Z} \\ 0 & \text{otherwise}\,. \end{cases} \tag{14b}$$

For example, for D_4, they read

$$(\mathcal{T}_{ij})_{i,j\geqslant 1} = \begin{pmatrix} x^1_0 & 0 & x^2_1 & -x^3_2 & 0 & -x^4_2 & -x^2_3 & 0 & -x^1_4 & 0 & -1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & x^1_2 & 0 & x^2_3 & -x^3_4 & 0 & -x^4_4 & -x^2_5 & 0 & -x^1_6 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & & & & \vdots & & & & & \ddots \end{pmatrix} \tag{15a}$$

$$(\mathcal{E}_{ij})_{i,j\geqslant 1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & x^4_0 & 0 & x^3_2 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & x^4_2 & 0 & x^3_4 & -1 & 0 & \\ & & & & & \vdots & & & & \ddots \end{pmatrix}. \tag{15b}$$

We keep the same notation (5), $\mathcal{T}_m(i, j, \pm x^a_k)$ $(1 \leqslant a \leqslant r-2)$ and $\mathcal{T}_m(i, j, -x^a_k)$, $\mathcal{E}_m(i, j, x^a_k)$ $(a = r-1, r)$ as in section 2. Our result in this section is as follows.

Theorem 4.1. For $m \in \mathbb{Z}_{\geqslant 1}$

$$T_m^{(a)}(u) = \det\big(\mathcal{T}_{2m-1}(1, 1, x^a_{-m+1}) + \mathcal{E}_{2m-1}(2, 3, x^r_{-m-r+a+4})\big) \qquad 1 \leqslant a \leqslant r-2 \qquad (16a)$$

$$T_m^{(r-1)}(u) = \mathrm{pf}\big(\mathcal{T}_{2m}(2, 1, -x^{r-1}_{-m+1}) + \mathcal{E}_{2m}(1, 2, x^{r-1}_{-m+1})\big) \qquad (16b)$$

$$T_m^{(r)}(u) = (-1)^m\, \mathrm{pf}\big(\mathcal{T}_{2m}(1, 2, -x^r_{-m+1}) + \mathcal{E}_{2m}(2, 1, x^r_{-m+1})\big) \qquad (16c)$$

solves the D_r T-system (3).

The matrices in (16*b*), (16*c*) are indeed antisymmetric. Equation (16*a*) is essentially conjecture (5.15) in [KNS1]. The Pfaffian formulae (16*b*), (16*c*) are new. By using them one can demonstrate the relations

$$T_m^{(r-1)}(u)T_m^{(r)}(u) = (-1)^m \det\big(\mathcal{T}_{2m}(1, 1, -x^{r-1}_{-m+1}) + \mathcal{E}_{2m}(2, 2, x^r_{-m+1})\big) \qquad (17a)$$

$$T_m^{(r-1)}(u+1)T_m^{(r)}(u-1) = (-1)^m \det\big(\mathcal{T}_{2m}(1, 1, -x^r_{-m}) + \mathcal{E}_{2m}(2, 2, x^{r-1}_{-m+2})\big) \qquad (17b)$$

$$T_{m+1}^{(r-1)}(u)T_m^{(r)}(u-1) = (-1)^{m+1} \det\big(\mathcal{T}_{2m+1}(1, 1, -x^{r-1}_{-m}) + \mathcal{E}_{2m+1}(2, 2, x^r_{-m})\big) \qquad (17c)$$

$$T_m^{(r-1)}(u+1)T_{m+1}^{(r)}(u) = (-1)^m \det\big(\mathcal{T}_{2m+1}(2, 1, x^{r-2}_{-m+1}) + \mathcal{E}_{2m+1}(1, 1, x^r_{-m})\big). \qquad (17d)$$

The proof of (17) is analogous to that of (13), which will be explained in the next section.

5. Proof of theorem 3.1

Here we shall outline the proof of theorem 3.1, namely that of the C_r T-system (2) starting from (12). As it turns out, all of the three-term relations in (2) reduce to Jacobi's identity:

$$D\begin{bmatrix}1\\1\end{bmatrix} D\begin{bmatrix}n\\n\end{bmatrix} = D\,D\begin{bmatrix}1, n\\1, n\end{bmatrix} + D\begin{bmatrix}1\\n\end{bmatrix} D\begin{bmatrix}n\\1\end{bmatrix}. \qquad (18)$$

Here D is the determinant of any n by n matrix and $D\big[\begin{smallmatrix} i_i, i_2, \ldots \\ j_1, j_2, \ldots\end{smallmatrix}\big]$ denotes its minor removing the i_kth rows and j_kth columns.

Let us prove equation (13*a*) first. Taking its square and substituting (12*b*), we must to show that

$$\det\big(\mathcal{T}_{2m}(1, 2, -x^r_{-m+\frac{1}{2}})\big) \det\big(\mathcal{T}_{2m}(1, 2, -x^r_{-m+\frac{3}{2}})\big) = \Big(\det\big(\mathcal{T}_{2m}(1, 1, -x^r_{-m+\frac{1}{2}})\big)\Big)^2. \qquad (19)$$

To see this we set

$$D = \det\big(\mathcal{T}_{2m+1}(1, 2, -x^r_{-m+\frac{1}{2}})\big) = \det\begin{pmatrix} 0 & -x^r_{-m+\frac{1}{2}} & -x^{r-1}_{-m+1} & \\ x^r_{-m+\frac{1}{2}} & 0 & -x^r_{-m+\frac{3}{2}} & \cdots \\ x^{r-1}_{-m+1} & x^r_{-m+\frac{3}{2}} & 0 & \\ & \vdots & & \ddots \end{pmatrix} = 0 \qquad (20)$$

since this is an antisymmetric matrix with odd size. From (20) it is easy to see that

$$\begin{aligned} D\begin{bmatrix}1\\1\end{bmatrix} &= \det\big(\mathcal{T}_{2m}(1, 2, -x^r_{-m+\frac{3}{2}})\big) \\ D\begin{bmatrix}2m+1\\2m+1\end{bmatrix} &= \det\big(\mathcal{T}_{2m}(1, 2, -x^r_{-m+\frac{1}{2}})\big) \\ D\begin{bmatrix}1\\2m+1\end{bmatrix} = D\begin{bmatrix}2m+1\\1\end{bmatrix} &= \det\big(\mathcal{T}_{2m}(1, 1, x^r_{-m+\frac{1}{2}})\big). \end{aligned} \qquad (21)$$

Thus equation (19) follows immediately from equations (21) and (18). In taking the square root of (19), the relative sign can be fixed by comparing the coefficients of $x^r_{-m+1/2}x^r_{-m+3/2}\cdots x^r_{m-1/2}$ on both sides, which agrees with (13*a*). Relation (13*b*) can be shown similarly by setting $D = \det\left(\mathcal{T}_{2m+2}(1,2,-x^r_{-m})\right)$.

Now we proceed to the proof of the T-system (2). To show (2*a*), it suffices to apply (18) for $D = \det\left(\mathcal{T}_{m+1}(1,1,x^a_{-\frac{m}{2}})\right) = T^{(a)}_{m+1}(u)$ and to note that $D\left[{1 \atop 1}\right] = T^{(a)}_m(u+\frac{1}{2})$, $D\left[{m+1 \atop m+1}\right] = T^{(a)}_m(u-\frac{1}{2})$, $D\left[{1,m+1 \atop 1,m+1}\right] = T^{(a)}_{m-1}(u)$, $D\left[{m+1 \atop 1}\right] = T^{(a+1)}_m(u)$ and $D\left[{1 \atop m+1}\right] = T^{(a-1)}_m(u)$. Similarly (2*b*) (equation (2*c*)) can be derived by setting $D = \det\left(\mathcal{T}_{2m+1}(1,1,x^{r-1}_{-m})\right) = T^{(r-1)}_{2m+1}(u)$ ($D = \det\left(\mathcal{T}_{2m+2}(1,1,x^{r-1}_{-m-\frac{1}{2}})\right) = T^{(r-1)}_{2m+2}(u)$) and using (13*a*) (equation 13*b*)) to identify $D\left[{2m+1 \atop 1}\right]$ with $T^{(r)}_m(u-\frac{1}{2})T^{(r)}_m(u+\frac{1}{2})$ ($T^{(r)}_m(u)T^{(r)}_{m+1}(u)$). Finally to show (2*d*), we put $D = \det\left(\mathcal{T}_{2m+1}(1,1,x^r_{-m})\right)$. Then from (12) and (13) we have

$$D = T^{(r)}_m(u)T^{(r)}_{m+1}(u) \qquad D\begin{bmatrix}1,2m+1\\1,2m+1\end{bmatrix} = T^{(r)}_{m-1}(u)T^{(r)}_m(u)$$
$$D\begin{bmatrix}1\\1\end{bmatrix} = T^{(r)}_m(u)T^{(r)}_m(u+1) \qquad D\begin{bmatrix}2m+1\\2m+1\end{bmatrix} = T^{(r)}_m(u-1)T^{(r)}_m(u) \tag{22}$$
$$D\begin{bmatrix}1\\2m+1\end{bmatrix} = T^{(r-1)}_{2m}(u) \qquad D\begin{bmatrix}2m+1\\1\end{bmatrix} = \left(T^{(r)}_m(u)\right)^2.$$

Substituting equation (22) in (18) (for $n = 2m+1$) and cancelling out the common factor $\left(T^{(r)}_m(u)\right)^2$, we obtain (2*d*). This completes the proof of theorem 3.1.

6. Summary and discussion

In this paper we have considered the difference equations (1), (2) and (3), which may be viewed as two-dimensional Toda equations on discrete spacetime as argued in (4). They have arisen as the B_r, C_r and D_r cases of the T-system, which are functional relations among commuting families of transfer matrices in the associated solvable lattice models. Under the initial condition $T^{(a)}_0(u) = 1$ ($1 \leqslant a \leqslant r$), we have given the solutions (9), (12) and (16) for $T^{(a)}_m(u)$ with $m \in \mathbb{Z}_{\geqslant 1}$. They are expressed in terms of Pfaffians or determinants of the matrices (6), (10) and (14), which contain only $\pm T^{(a)}_1(u + \text{shift})$ or ± 1 as their matrix elements. This confirms the earlier conjectures [KNS1] and extends them to the full solutions.

It will be interesting to extend a similar analysis to the T-system for the exceptional algebras $E_{6,7,8}$, F_4, G_2 [KNS1] and also the twisted quantum affine algebras $A^{(2)}_n$, $D^{(2)}_n$, $E^{(2)}_6$ and $D^{(3)}_4$ [KS]. In fact, the solutions to the $A^{(2)}_n$, $D^{(2)}_n$ and $D^{(3)}_4$ cases can be obtained just by imposing the 'modulo σ relations' (equations (3.4) in [KS]) on the corresponding non-twisted cases A_n, D_n and D_4 treated in this paper. On the other hand, to deal with the exceptional cases, it seems necessary to introduce matrices whose elements are some higher-order expressions in the $T^{(a)}_1(u)$ analogous to [KOS].

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